

# SOME CONSEQUENCES OF VON NEUMANN ALGEBRA UNIQUENESS

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ABSTRACT. In this note, we derive some consequences of the von Neumann algebra uniqueness theorems developed in the previous paper [3]. In particular,

- (1) we solve a question raised in [7], by proving that if  $\mathcal{A}$  is a separable simple nuclear  $C^*$ -algebra and  $\pi_i$ ,  $i = 1, 2$ , are type III representations of  $A$  on a separable Hilbert space, then for  $\pi_1$  and  $\pi_2$  being algebraically equivalent, it is necessary and sufficient that there is an automorphism  $\alpha$  of  $A$  such that  $\pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent.
- (2) we give a new (short) proof of the equivalence of injectivity and extreme amenability (of the corresponding unitary group) for countably decomposable properly infinite von Neumann algebras.
- (3) using ideas of [15], we show that the Connes embedding problem is equivalent to many topological groups having the Kirchberg property.

## 1. INTRODUCTION

To prove in his famous 1967 paper that the now called Powers factors  $R_\lambda$ ,  $0 < \lambda < 1$ , are non isomorphic, R.T. Powers [16] proved that if  $A$  is a UHF  $C^*$ -algebra and  $\pi_i$ ,  $i = 1, 2$ , are representations of  $A$  on a separable Hilbert space, then for  $\pi_1$  and  $\pi_2$  being algebraically equivalent (i.e., the von Neumann algebras generated by  $\pi_1(A)$  and  $\pi_2(A)$  are isomorphic), it is necessary and sufficient that there is an automorphism  $\alpha$  of  $A$  such that  $\pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent.

Powers' characterization was generalized either to a larger class of  $C^*$ -algebras  $A$  or to special classes of representations (see [2], [13], [11], [12]). In the first section of this paper, we resolve a question raised in [7] by showing that

**Theorem 1.1.** *Let  $\mathcal{A}$  be a separable simple nuclear  $C^*$ -algebra, and let  $\pi_1$  and  $\pi_2$  be nondegenerate type III representations of  $\mathcal{A}$  on a separable Hilbert space. Then  $\pi_1$  and  $\pi_2$  are algebraically equivalent if and only if there exists  $\alpha \in \text{AI}nn(\mathcal{A})$  such that  $\pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent.*

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The main step of the proof of this theorem is a consequence of [3] Corollary 3.5.

In Section 3, we use Voiculescu-type uniqueness theorems developed in [3] to study extreme amenability. Recall that a topological group  $G$  is said to be *extremely amenable* if every continuous action of  $G$ , on a compact Hausdorff topological space, has a fixed point and that no locally compact group is extremely amenable [17]. However, large classes of interesting “massive” (i.e., non locally compact) topological groups have been shown to be extremely amenable. A basic example is Gromov and Milman’s result that for a separable infinite dimensional Hilbert space  $\mathcal{H}$ , the unitary group  $U(\mathcal{H})$ , given the weak\* topology, is extremely amenable [9]. Subsequently it was shown in [8] that many naturally occurring nonlocally compact groups - coming from operator algebras, dynamical systems and combinatorics are extremely amenable. We give in Theorem 3.3 another proof of the following result, originally proven in [8].

**Theorem 1.2.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite injective von Neumann algebra. Let  $U(\mathcal{M})$  be the unitary group of  $\mathcal{M}$ , given the  $\sigma$ -strong\* topology.*

*Then  $\mathcal{M}$  is injective if and only if  $U(\mathcal{M})$  is extremely amenable.*

Finally we use our techniques to study the Kirchberg property and the Connes embedding problem. The definitions are recalled in Section 4. In [15], Pestov and Uspenskij observed that an affirmative answer to the Connes embedding problem is equivalent to  $U(\mathcal{H})$  having the Kirchberg property. We generalize their result by showing that:

**Theorem 1.3.** *The following statements are equivalent:*

- (1) *The Connes embedding problem has an affirmative answer.*
- (2) *For every unital separable simple nuclear  $C^*$ -algebra  $\mathcal{A}$ ,  $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$  has the Kirchberg property.*
- (3) *There exists a unital separable simple nuclear  $C^*$ -algebra  $\mathcal{A}$  such that  $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$  has the Kirchberg property.*
- (4) *For every countably decomposable injective properly infinite von Neumann algebra  $\mathcal{M}$ ,  $U(\mathcal{M})$  has the Kirchberg property.*
- (5) *There exists an injective properly infinite von Neumann algebra  $\mathcal{M}$  with separable predual such that  $U(\mathcal{M})$  has the Kirchberg property.*

## 2. QUASI-EQUIVALENT REPRESENTATIONS

In [7], Futamura, Kataoka and Kishimoto prove that if  $A$  belongs to a restricted class of simple, separable, nuclear  $C^*$ -algebras, and if  $\pi_1$  and  $\pi_2$  are type *III* representations of  $A$  on a separable Hilbert space, then they are algebraically equivalent if and only if there is an asymptotically inner automorphism  $\alpha$  of  $A$  such that  $\pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent.

In this section, we show, by modifying our previous results, that Futamura, Kataoka and Kishimoto's statement is true for all simple, separable, nuclear  $C^*$ -algebras. Our result also generalizes results of Powers [16], of Bratteli [2], of Kishimoto, Ozawa, and Sakai [13] and of Kishimoto [11], [12]. As indicated in the introduction, Powers [16] was interested in constructing a continuum of nonisomorphic type III factors, and Bratteli [2] in representations of AF-algebras, as well as the problem of when the pure state space is homogeneous (see also [13]). Kishimoto [11] was interested in representations that were covariant with respect to an approximately inner flow. We refer the reader to these papers and their references for details as well as basic definitions and background results.

The following proposition is an immediate consequence of [3] Corollary 3.5 and is the starting point of our proof. Before stating it, let us recall the notion of a *full*  $*$ -homomorphism:

**Definition 2.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $C^*$ -algebras. Then a  $*$ -homomorphism  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  is said to be full if for every nonzero positive element  $a \in \mathcal{A}$ , its image  $\phi(a)$  is a full element of  $\mathcal{B}$  (i.e.,  $\phi(a)$  is not contained in any proper  $C^*$ -algebra ideal of  $\mathcal{B}$ ).*

Note that a full  $*$ -homomorphism is necessarily injective and that conversely an injective  $*$ -homomorphism from a  $C^*$ -algebra to a type III and countably decomposable von Neumann factor is full.

**Proposition 2.1.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite von Neumann algebra, and let  $\mathcal{A}$  be a nuclear  $C^*$ -algebra. Let  $\phi, \psi : \mathcal{A} \rightarrow \mathcal{M}$  be two full  $*$ -homomorphisms.*

*Then there exists a net  $\{v_\alpha\}$  of partial isometries in  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ ,*

$$\|v_\alpha^* \phi(a) v_\alpha - \psi(a)\| \rightarrow 0$$

The next three lemmas generalize arguments from [5] Corollary II.5.4, II.5.5 and II.5.6.

**Lemma 2.2.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite von Neumann algebra, and let  $\mathcal{A}$  be a full nuclear  $C^*$ -subalgebra of  $\mathcal{M}$  with  $1_{\mathcal{M}} \in \mathcal{A}$ . Let  $\rho : \mathcal{A} \rightarrow \mathcal{M}$  be a full  $*$ -homomorphism such that  $\rho(1_{\mathcal{M}}) = 1_{\mathcal{M}}$ .*

*Then there exists a net  $\{v_\alpha\}$  of isometries of  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ .*

$$\|v_\alpha \rho(a) - a v_\alpha\| \rightarrow 0$$

*Proof.* By Proposition 2.1, there exists a net  $\{v_\alpha\}$  of partial isometries in  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ ,

$$(2.1) \quad \|\rho(a) - v_\alpha^* a v_\alpha\| \rightarrow 0$$

Note that since  $1_{\mathcal{M}} \in \mathcal{A}$  and  $\rho(1_{\mathcal{M}}) = 1_{\mathcal{M}}$ , equation (2.1) implies that for every  $\epsilon > 0$ ,  $\|1_{\mathcal{M}} - v_{\alpha}^* v_{\alpha}\| < \epsilon$ , for sufficiently “large”  $\alpha$ . Hence, we may assume that for all  $\alpha$ ,  $v_{\alpha}$  is an isometry in  $\mathcal{M}$ .

Hence, for all  $a \in \mathcal{A}$ ,

$$\begin{aligned}
& (v_{\alpha}\rho(a) - av_{\alpha})^*(v_{\alpha}\rho(a) - av_{\alpha}) \\
&= (\rho(a^*)v_{\alpha}^* - v_{\alpha}^*a^*)(v_{\alpha}\rho(a) - av_{\alpha}) \\
&= \rho(a^*)\rho(a) - \rho(a^*)v_{\alpha}^*av_{\alpha} - v_{\alpha}^*a^*v_{\alpha}\rho(a) + v_{\alpha}^*a^*av_{\alpha} \\
&= \rho(a^*)(\rho(a) - v_{\alpha}^*av_{\alpha}) + (\rho(a^*) - v_{\alpha}^*a^*v_{\alpha})\rho(a) + (v_{\alpha}^*a^*av_{\alpha} - \rho(a^*a)) \\
&\rightarrow 0
\end{aligned}$$

□

Let  $\mathcal{A}$  be a  $C^*$ -algebra, and let  $\mathcal{M}$  be a von Neumann algebra. We say that a  $*$ -homomorphism  $\rho : \mathcal{A} \rightarrow \mathcal{M}$  is *nondegenerate* if  $1_{\overline{\rho(\mathcal{A})}^{weak*}} = 1_{\mathcal{M}}$ . If  $\mathcal{A}$  is a  $C^*$ -subalgebra of  $\mathcal{M}$  then  $\mathcal{A}$  is a *nondegenerate*  $C^*$ -subalgebra if the inclusion map  $\mathcal{A} \rightarrow \mathcal{M}$  is nondegenerate.

**Lemma 2.3.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite von Neumann algebra. Let  $\mathcal{A} \subseteq \mathcal{M}$  be a full nondegenerate nuclear  $C^*$ -subalgebra and let  $\rho : \mathcal{A} \rightarrow \mathcal{M}$  be a full nondegenerate  $*$ -homomorphism.*

*Let  $S_1, S_2 \in \mathcal{M}$  be elements such that  $S_1^*S_1 = S_2^*S_2 = 1_{\mathcal{M}}$  and  $S_1S_1^* + S_2S_2^* = 1_{\mathcal{M}}$ , and consider the full nondegenerate  $*$ -homomorphism*

$$\mathcal{A} \rightarrow \mathcal{M} : a \mapsto S_1aS_1^* + S_2\rho(a)S_2^*.$$

*Then for every  $\epsilon > 0$ , for every finite subset  $\mathcal{F} \subset \mathcal{M}$ , there exists a unitary  $w \in \mathcal{M}$  such that*

$$\|w(S_1aS_1^* + S_2\rho(a)S_2^*)w^* - a\| < \epsilon$$

*for all  $a \in \mathcal{F}$ .*

*Proof.* Firstly, we may assume that  $\mathcal{A}$  is unital (i.e.,  $1_{\mathcal{M}} \in \mathcal{A}$ ) and  $\rho(1_{\mathcal{M}}) = 1_{\mathcal{M}}$ . (For otherwise, we can replace  $\mathcal{A}$  with its unitization  $\mathcal{A} + \mathbb{C}1_{\mathcal{M}}$  (which will still be nuclear) and replace  $\rho$  by the (unitized) map  $a + \alpha 1_{\mathcal{M}} \mapsto \rho(a) + \alpha 1_{\mathcal{M}}$ . Here is a rough sketch of the proof that the unitizations are still full: It suffices to prove that if  $a \in \mathcal{A}$  is nonzero self-adjoint and  $\alpha \geq 0$  is such that  $a + \alpha 1 \in (\mathcal{A} + \mathbb{C}1)_+$ , then  $\rho(a) + \alpha 1$  is a full element of  $\mathcal{M}$ . To do this, find a nonzero element  $c \in \mathcal{A}$  with  $0 \leq c \leq a + \alpha 1$ . Then  $\rho(c)$  is a full element of  $\mathcal{M}$  and  $0 \leq \rho(c) \leq \rho(a) + \alpha 1$ . Hence,  $\rho(a) + \alpha 1$  is a full element of  $\mathcal{M}$ .)

Let  $\epsilon > 0$  and a finite subset  $\mathcal{F} \subset \mathcal{A}$  be given. We may assume that  $\mathcal{F}$  is self-adjoint; i.e., for all  $a \in \mathcal{A}$ , if  $a \in \mathcal{F}$  then  $a^* \in \mathcal{F}$ .

Let  $\lambda : \mathcal{A} \rightarrow \mathcal{M}$  be given by  $\lambda(a) =_{df} S_2\rho(a)S_2^*$  for all  $a \in \mathcal{A}$ , and let  $q =_{df} \lambda(1) = S_2S_2^*$ .

Let  $\{j_n\}_{n=1}^{\infty}$  be a sequence of partial isometries in  $\mathcal{M}$  such that  $j_n^*j_n = q$  for all  $n \geq 1$ ,  $j_1 = j_1^* = q$ , and  $\sum_{n=1}^{\infty} j_n j_n^* = 1$ .

Set  $T =_{df} \sum_{n=1}^{\infty} j_n j_{n+1}^* \in \mathcal{M}$ . Then

$$T^*T = \sum_{n,m} j_{n+1} j_n^* j_m j_{m+1}^* = \sum_{n=1}^{\infty} j_{n+1} j_n^* j_n j_{n+1}^* = 1 - q$$

and

$$TT^* = \sum_{n,m} j_n j_{n+1}^* j_{m+1} j_m^* = \sum_{n=1}^{\infty} j_n j_{n+1}^* j_{n+1} j_n^* = 1.$$

(Hence,  $T^*$  is an isometry.)

Set  $\lambda^{(\infty)} : \mathcal{A} \rightarrow \mathcal{M}$  by  $\lambda^{(\infty)}(a) =_{df} \sum_{n=1}^{\infty} j_n \lambda(a) j_n^*$  for all  $a \in \mathcal{A}$ . Hence,  $\lambda^{(\infty)}(1) = \sum_{n=1}^{\infty} j_n q j_n^* = 1$ . Hence,  $\lambda^{(\infty)}$  is a full unital  $*$ -homomorphism.

By Lemma 2.2, let  $v \in \mathcal{M}$  be an isometry such that

$$(2.2) \quad \|v \lambda^{(\infty)}(a) - av\| < \epsilon/5$$

for all  $a \in \mathcal{F}$ .

Now let

$$w =_{df} (1 - vv^* + vT^*v^*)T + vj_1 \in \mathcal{M}.$$

One can check that  $w$  is a unitary of  $\mathcal{M}$ .

Note that since  $T^*$  is an isometry with  $TT^* = 1 - q$ , we may assume that  $S_1 = T^* = \sum_{n=1}^{\infty} j_{n+1} j_n^*$ .

Hence, one can check that for all  $a \in \mathcal{A}$ ,

$$\begin{aligned} & aw - w(S_1 a S_1^* + S_2 \rho(a) S_2^*) \\ &= aw - w(T^* a T + \lambda(a)) \\ &= (vv^* a - avv^*)T + (avT^*v^* - vT^*v^*a)T + (avj_1 - vj_1\lambda(a)) \end{aligned}$$

Also, one can check that for all  $a \in \mathcal{A}$ , we have the following:

- (1)  $(vv^*a - avv^*)T = [v(a^*v - v\lambda^{(\infty)}(a^*))^* - (av - v\lambda^{(\infty)}(a))v^*]T$ .
- (2)  $(avT^*v^* - vT^*v^*a)T = [(av - v\lambda^{(\infty)}(a))T^*v^* - vT^*(a^*v - v\lambda^{(\infty)}(a^*))^*]T$   
(Note that  $T^*\lambda^{(\infty)}(a) = \lambda^{(\infty)}(a)T^*$  for all  $a \in \mathcal{A}$ .)
- (3)  $avj_1 - vj_1\lambda(a) = (av - v\lambda^{(\infty)}(a))j_1$

From the above and from (2.2), we have that for all  $a \in \mathcal{F}$ ,

$$\|aw - w(S_1 a S_1^* + S_2 \rho(a) S_2^*)\| < \epsilon.$$

Hence, for all  $a \in \mathcal{F}$ ,

$$\|a - w(S_1 a S_1^* + S_2 \rho(a) S_2^*)w^*\| < \epsilon$$

as required.  $\square$

**Lemma 2.4.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite von Neumann algebra and let  $\mathcal{A}$  be a nuclear  $C^*$ -algebra. Let  $\phi, \psi : \mathcal{A} \rightarrow \mathcal{M}$  be full nondegenerate  $*$ -homomorphisms, and let  $S_1, S_2 \in \mathcal{M}$  be elements with  $S_i^* S_i = 1$  ( $i = 1, 2$ ) and  $S_1 S_1^* + S_2 S_2^* = 1$ .*

Then  $\psi$  and  $Ad(S_1)\psi + Ad(S_2)\phi$  are (norm-) approximately unitarily equivalent. I.e., there exists a net  $\{u_\alpha\}$  of unitaries in  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ ,

$$\|u_\alpha^* \psi(a) u_\alpha - (S_1 \psi(a) S_1^* + S_2 \phi(a) S_2^*)\| \rightarrow 0$$

*Proof.* Since  $\psi$  is full and nondegenerate, we can identify  $\mathcal{A}$  with  $\psi(\mathcal{A}) \subset \mathcal{M}$ , view  $\mathcal{A}$  as a full nondegenerate  $C^*$ -subalgebra of  $\mathcal{M}$ , and replace  $\psi$  with the inclusion map. The result then follows from Lemma 2.3 (where we take  $\rho = \phi$ ).  $\square$

**Proposition 2.5.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite von Neumann algebra and let  $\mathcal{A}$  be a nuclear  $C^*$ -algebra. Let  $\phi, \psi : \mathcal{A} \rightarrow \mathcal{M}$  be full nondegenerate  $*$ -homomorphisms.*

*Then  $\phi$  and  $\psi$  are approximately unitarily equivalent in norm. I.e., there exists a net  $\{u_\alpha\}$  of unitaries in  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ ,*

$$\|u_\alpha^* \phi(a) u_\alpha - \psi(a)\| \rightarrow 0$$

*Proof.* Let  $S_1, S_2 \in \mathcal{M}$  be isometries with  $S_1 S_1^* + S_2 S_2^* = 1$ . By applying Lemma 2.4 twice, we have that  $\phi$  and  $Ad(S_1)\psi + Ad(S_2)\phi$  are (norm-) approximately unitarily equivalent, and  $\psi$  and  $Ad(S_1)\psi + Ad(S_2)\phi$  are (norm-) approximately unitarily equivalent. Hence,  $\phi$  and  $\psi$  are (norm-) approximately unitarily equivalent, as required.  $\square$

As a corollary of Proposition 2.5, we get the following:

**Corollary 2.6.** *Let  $\mathcal{M}$  be a countably generated von Neumann, and let  $\mathcal{A}$  be a nontype I simple nuclear  $C^*$ -algebra. Suppose that  $\phi, \psi : \mathcal{A} \rightarrow \mathcal{M}$  are nondegenerate  $*$ -homomorphisms. Suppose also that either*

- (1)  $\mathcal{A}$  is purely infinite, or
- (2)  $\mathcal{M}$  has no type II summand, or
- (3)  $\mathcal{A}$  is unital and  $\mathcal{M}$  has no type  $II_1$  summand.

*Then  $\phi$  and  $\psi$  are both (norm-) approximately unitarily equivalent. I.e., there exists a net  $\{u_\alpha\}$  of unitaries in  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ ,*

$$\|u_\alpha^* \phi(a) u_\alpha - \psi(a)\| \rightarrow 0$$

*Proof.* Firstly, since  $\mathcal{M}$  is countably generated, it is a (possibly infinite) direct product of von Neumann algebras with separable predual. Hence, we may assume that  $\mathcal{M}$  has separable predual.

Decompose  $\mathcal{M}$  into a direct sum  $\mathcal{M} = \mathcal{M}_\infty \oplus \mathcal{M}_{II_1} \oplus \mathcal{M}_{I_f}$  where  $\mathcal{M}_\infty$  is properly infinite,  $\mathcal{M}_{II_1}$  is type  $II_1$  and  $\mathcal{M}_{I_f}$  is finite type I.

Since  $\mathcal{A}$  is nontype I and simple, and since we have a nondegenerate  $*$ -homomorphism  $\phi : \mathcal{A} \rightarrow \mathcal{M}$ ,  $\mathcal{M}_{I_f} = 0$ . Hence,

$$\mathcal{M} = \mathcal{M}_\infty \oplus \mathcal{M}_{II_1}.$$

Let us first assume (1). Since  $\mathcal{A}$  is simple purely infinite and since  $\phi : \mathcal{A} \rightarrow \mathcal{M}$  is a nondegenerate  $*$ -homomorphism,  $\mathcal{M}_{II_1} = 0$ . Hence,  $\mathcal{M} = \mathcal{M}_\infty$ ; i.e.,  $\mathcal{M}$  is properly infinite. Since  $\mathcal{A}$  is simple purely infinite, let  $p \in \mathcal{A}$  be a nonzero properly infinite projection. Hence,  $\phi(p)$  and  $\psi(p)$  are properly infinite projections in  $\mathcal{M}$ . Since  $\mathcal{A}$  is simple and  $\phi, \psi$  are nondegenerate, we have, by [10] Corollary 6.3.5, that  $\phi(p)$  and  $\psi(p)$  are both Murray-von Neumann equivalent to  $1_{\mathcal{M}}$ . Hence, since  $\mathcal{A}$  is simple,  $\phi$  and  $\psi$  are full. Hence, by Proposition 2.5,  $\phi$  and  $\psi$  are (norm-) approximately unitarily equivalent.

Next, assume (2). Since  $\mathcal{M}$  has not type  $II$  summand,  $\mathcal{M}_{II_1} = 0$  and  $\mathcal{M} = \mathcal{M}_\infty$  is properly infinite. Indeed,  $\mathcal{M}$  is a type III von Neumann algebra with separable predual. Let  $a, b \in \mathcal{A}$  be nonzero positive elements such that  $0 \leq a \leq b$  and  $ab = a$ . Hence, there exist (necessarily nonzero) projections  $p, q \in \mathcal{M}$  such that  $\phi(a) \leq p \leq \phi(b)$  and  $\psi(a) \leq q \leq \psi(b)$ . (E.g., take  $p, q$  to be the support projections of  $\phi(a), \psi(a)$  respectively.) Since  $\phi(a) \neq 0$ ,  $\psi(a) \neq 0$ , and since  $\mathcal{M}$  is type III,  $p, q$  must be properly infinite projections in  $\mathcal{M}$  (e.g., see [10] Proposition 6.3.7). Since  $\phi, \psi$  are nondegenerate and since  $\phi(a) \leq p$  and  $\psi(a) \leq q$ ,  $p, q$  both must have central carrier 1. Hence, by [10] Corollary 6.3.5,  $p, q$  are both Murray-von Neumann equivalent to 1. Hence, since  $p \leq \phi(b)$  and  $q \leq \psi(b)$ , and since  $\mathcal{A}$  is simple,  $\phi, \psi$  are full  $*$ -homomorphisms. Hence, by Proposition 2.5,  $\phi$  and  $\psi$  are (norm-) approximately unitarily equivalent, as required.

Finally, assume (2). Since  $\mathcal{M}$  does not contain a type  $II_1$  direct summand,  $\mathcal{M} = \mathcal{M}_\infty$ ; i.e.,  $\mathcal{M}$  is properly infinite. Since  $\mathcal{A}$  is unital and since  $\phi, \psi$  are nondegenerate,  $\phi(1_{\mathcal{A}}) = \psi(1_{\mathcal{A}}) = 1_{\mathcal{M}}$ . Hence, since  $\mathcal{A}$  is simple,  $\phi, \psi$  are full  $*$ -homomorphisms. Hence, by Proposition 2.5,  $\phi$  and  $\psi$  are (norm-) approximately unitarily equivalent. This completes the final case.  $\square$

Using Corollary 2.6, we have the following result, which generalizes [7] Corollaries 3.6 and 3.8.

**Corollary 2.7.** *Let  $\mathcal{A}$  be a simple separable nuclear  $C^*$ -algebra, and let  $\pi_1$  and  $\pi_2$  be nondegenerate representations of  $\mathcal{A}$  on a separable Hilbert space  $\mathcal{H}$  such that  $\pi_1(\mathcal{A})'' = \pi_2(\mathcal{A})'' = \mathcal{M}$ . Suppose that either*

- (1)  $\mathcal{A}$  is purely infinite or
- (2)  $\mathcal{M}$  does not contain a type  $II$  direct summand or
- (3)  $\mathcal{A}$  is unital and  $\mathcal{M}$  does not contain a type  $II_1$  direct summand.

*Then there exists a sequence  $\{u_n\}$  of unitaries in  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ ,*

$$\|u_n^* \pi_1(a) u_n - \pi_2(a)\| \rightarrow 0$$

*Proof.* If  $\mathcal{A}$  is isomorphic to the algebra of compact operators on a separable Hilbert space, then the result follows from [5] Corollary I.10.6 and Theorem I.10.7. Hence, we may assume that  $\mathcal{A}$  is not type I.

If  $\mathcal{A}$  is nontype I then the result follows from Corollary 2.6.  $\square$

Let  $\mathcal{A}$  be a  $C^*$ -algebra. Let  $\pi_1$  and  $\pi_2$  be nondegenerate representations of  $\mathcal{A}$  on Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively. Recall that  $\pi_1$  and  $\pi_2$  are said to be *algebraically equivalent* if  $\pi_1(\mathcal{A})''$  and  $\pi_2(\mathcal{A})''$  are isomorphic as von Neumann algebras. Recall that  $\pi_1$  and  $\pi_2$  are said to be *quasi-equivalent* if the map  $\pi_1(a) \mapsto \pi_2(a)$  (for  $a \in \mathcal{A}$ ) extends to an isomorphism of  $\pi_1(\mathcal{A})''$  onto  $\pi_2(\mathcal{A})''$ .

Next, recall that a  $*$ -automorphism  $\alpha \in \text{Aut}(\mathcal{A})$  is said to be *asymptotically inner* if there is a continuous path  $\{u_t\}_{t \in [0, \infty)}$  of unitaries in  $\mathcal{A}$  (or  $\mathcal{A} + \mathbb{C}1$  if  $\mathcal{A}$  is nonunital) such that  $\lim_{t \rightarrow \infty} u_t^* a u_t = \alpha(a)$  for all  $a \in \mathcal{A}$ . We let  $\text{AI}nn(\mathcal{A})$  denote the group of asymptotically inner automorphisms of  $\mathcal{A}$ .

The next result generalizes [7] Corollary 4.3 and part of Corollary 4.4.

**Theorem 2.8.** *Let  $\mathcal{A}$  be simple separable nuclear  $C^*$ -algebra. Let  $\pi_1$  and  $\pi_2$  be nondegenerate representations of  $\mathcal{A}$  such that  $\pi_1(\mathcal{A})''$  (or  $\pi_2(\mathcal{A})''$ ) has separable predual. Suppose that one of the following conditions hold:*

- (1)  $\mathcal{A}$  is purely infinite.
- (2)  $\pi_1(\mathcal{A})''$  (or  $\pi_2(\mathcal{A})''$ ) contains no type II summand
- (3)  $\mathcal{A}$  is unital and  $\pi_1(\mathcal{A})''$  (or  $\pi_2(\mathcal{A})''$ ) does not contain a type  $II_1$  summand.

*Then  $\pi_1$  and  $\pi_2$  are algebraically equivalent if and only if there exists an  $\alpha \in \text{AI}nn(\mathcal{A})$  such that  $\pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent.*

*Proof.* The “if” direction is clear.

We now prove the “only if” direction.

Case 1:  $\pi_1(\mathcal{A})'' = \pi_2(\mathcal{A})''$ .

In this case, the result follows from Corollary 2.7 and [7] Theorem 4.1.

Case 2: General case.

Let  $\Phi : \pi_1(\mathcal{A})'' \rightarrow \pi_2(\mathcal{A})''$  be an isomorphism. Consider the nondegenerate representation  $\Phi \circ \pi_1$  of  $\mathcal{A}$ . Since  $\Phi$  is normal and by hypothesis,  $(\Phi \circ \pi_1)(\mathcal{A})'' = \Phi(\pi_1(\mathcal{A})'') = \pi_2(\mathcal{A})''$ . Hence, by Case 1, there exists  $\alpha \in \text{AI}nn(\mathcal{A})$  such that  $\Phi \circ \pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent. Since  $\pi_1 \circ \alpha$  and  $\Phi \circ \pi_1 \circ \alpha$  are quasi-equivalent,  $\pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent.  $\square$

As a consequence of the above result, we resolve a question in [7]:

**Theorem 2.9.** *Let  $\mathcal{A}$  be a separable simple nuclear  $C^*$ -algebra, and let  $\pi_1$  and  $\pi_2$  be nondegenerate type III representations of  $\mathcal{A}$  on a separable Hilbert space. Then  $\pi_1$  and  $\pi_2$  are algebraically equivalent if and only if there exists  $\alpha \in \text{AI}nn(\mathcal{A})$  such that  $\pi_1 \circ \alpha$  and  $\pi_2$  are quasi-equivalent.*

### 3. EXTREME AMENABILITY

A (Hausdorff) topological group  $G$  is said to be *extremely amenable* if every continuous action of  $G$  on a compact Hausdorff space has a fixed point.

If  $\mathcal{M}$  is an injective von Neumann algebra with no type I summand then its unitary group  $U(\mathcal{M})$  with the  $\sigma$ -strong\* topology is extremely amenable



(see [8] Theorem 3.3 and Corollary 3.6). We refer the reader to [8] for the background and basic definitions in the subject.

In this section, if  $\mathcal{M}$  is a countably decomposable properly infinite injective von Neumann algebra, we use our uniqueness results from [3] to give another proof of the extreme amenability of  $U(\mathcal{M})$ .

Let  $\mathcal{A}, \mathcal{C}$  be  $C^*$ -algebras. Then a  $*$ -homomorphism  $\rho : \mathcal{A} \rightarrow \mathcal{C}$  is said to be *full* if for every nonzero positive element  $a \in \mathcal{A}$ ,  $\rho(a)$  is a full element of  $\mathcal{C}$ . If  $\mathcal{A}$  is a  $C^*$ -subalgebra of  $\mathcal{C}$ , then  $\mathcal{A}$  is said to be a *full  $C^*$ -subalgebra* of  $\mathcal{C}$  if the inclusion map  $\mathcal{A} \subseteq \mathcal{C}$  is full.

We will be using the following result, which is a consequence of [3] Proposition 4.3 and Lemma 3.9.

**Proposition 3.1.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite injective von Neumann algebra, and let  $\mathcal{A}$  be a unital  $C^*$ -algebra. Suppose that  $\phi, \psi : \mathcal{A} \rightarrow \mathcal{M}$  are two unital full  $*$ -homomorphisms.*

*Then there exists a net  $\{u_\alpha\}$  of unitaries in  $\mathcal{M}$  such that for all  $a \in \mathcal{A}$ ,*

$$u_\alpha \phi(a) u_\alpha^* \rightarrow \psi(a)$$

*in the  $\sigma$ -strong\* topology.*

Following the arguments of [14], we will use the notion of a locally dense net.

**Definition 3.1.** *Let  $G$  be a topological group, and let  $\Pi$  be a uniform structure on  $G$  that is compatible with the topology on  $G$ . Let us say that a net of subgroups  $\{H_\lambda\}_{\lambda \in \Lambda}$  of  $G$  is *locally dense in  $G$*  if for every entourage  $\mathcal{E} \in \Pi$ , for every finite set  $\{g_i\}_{i=1}^n$  in  $G$ , there is a  $\lambda_0 \in \Lambda$  such that for  $\lambda \geq \lambda_0$  and for  $1 \leq i \leq n$ ,  $(\{g_i\} \times H_\lambda) \cap \mathcal{E} \neq \emptyset$ .*

The next lemma is an exercise in point set topology.

**Lemma 3.2.** *Let  $G$  be a topological group, and let  $\Pi$  be a uniform structure on  $G$  that is compatible with the topology on  $G$ . If  $(G, \Pi)$  has a locally dense net  $\{H_\lambda\}_{\lambda \in \Lambda}$  consisting of extremely amenable subgroups, then  $G$  is extremely amenable.*

We now fix a notation. For a von Neumann algebra  $\mathcal{M}$  and a normal positive linear functional  $\chi \in \mathcal{M}_*$ , let  $\|\cdot\|_\chi^\sharp$  be the seminorm on  $\mathcal{M}$  that is defined by  $\|x\|_\chi^\sharp =_{df} \sqrt{\chi(x^*x) + \chi(xx^*)}$  for all  $x \in \mathcal{M}$ .

**Theorem 3.3.** *Let  $\mathcal{M}$  be a countably decomposable properly infinite injective von Neumann algebra. Let  $U(\mathcal{M})$  be the unitary group of  $\mathcal{M}$ , given the  $\sigma$ -strong\* topology.*

*Then  $\mathcal{M}$  is injective if and only if  $U(\mathcal{M})$  is extremely amenable.*

*Proof.* The “if” direction follows from [4] and the fact that extreme amenability implies amenability.

We now prove the “only if” direction.

Since  $\mathcal{M}$  is countably decomposable, let  $\chi \in \mathcal{M}_*$  be a faithful normal state on  $\mathcal{M}$ . Then the  $\sigma$ -strong\* topology on  $U(\mathcal{M})$  is given by the metric induced by the norm  $\|\cdot\|_\chi^\sharp$  (which gives a uniform structure on  $U(\mathcal{M})$ ).

Let  $\epsilon > 0$  be given and let  $x_1, x_2, \dots, x_m$  be a finite subset of  $U(\mathcal{M})$ .

Now since  $\mathcal{M}$  is properly infinite, let  $\mathcal{N}$  be a von Neumann algebra and  $\mathcal{H}$  a separable infinite dimensional Hilbert space such that  $\mathcal{M} \cong \mathcal{N} \otimes \mathbb{B}(\mathcal{H})$ .

Let  $\mathcal{A}$  be the unital C\*-subalgebra of  $\mathcal{M}$  that is generated by  $\{x_1, x_2, \dots, x_m\}$ .

Now let  $\psi : \mathcal{A} \rightarrow 1_{\mathcal{N}} \otimes \mathbb{B}(\mathcal{H}) \subset \mathcal{M}$  be a full unital \*-homomorphism. Since  $\mathcal{M}$  is properly infinite, let  $S_1, S_2 \in \mathcal{M}$  be isometries with orthogonal ranges with  $S_1 S_1^* + S_2 S_2^* = 1_{\mathcal{M}}$  such that

$$\|x_k - (S_1 x_k S_1^* + S_2 \psi(x_k) S_2^*)\|_\chi^\sharp < \epsilon/3$$

for  $1 \leq k \leq m$

Let  $\phi : \mathcal{A} \rightarrow \mathcal{M}$  be the full unital \*-homomorphism that is given by  $\phi(a) =_{df} S_1 a S_1^* + S_2 \psi(a) S_2^*$  for all  $a \in \mathcal{A}$ . Hence,

$$(3.1) \quad \|x_k - \phi(x_k)\|_\chi^\sharp < \epsilon/3$$

for  $1 \leq k \leq m$ .

By Proposition 3.1, let  $u \in \mathcal{M}$  be a unitary such that

$$\|\phi(x_k) - u \psi(x_k) u^*\|_\chi^\sharp < \epsilon/3$$

for  $1 \leq k \leq m$ . From this and (3.1),

$$\|x_k - u \psi(x_k) u^*\|_\chi^\sharp < \epsilon$$

for  $1 \leq k \leq m$ .

By [9], the unitary group of  $u(1_{\mathcal{N}} \otimes \mathbb{B}(\mathcal{H}))u^* \cong \mathbb{B}(\mathcal{H})$ , given the  $\sigma$ -strong\* topology, is extremely amenable. Hence, by Lemma 3.2, since  $\epsilon$  and  $x_1, x_2, \dots, x_m$  are arbitrary (so we can construct an appropriate locally dense net),  $U(\mathcal{M})$  is extremely amenable.  $\square$

#### 4. THE KIRCHBERG PROPERTY

If  $G$  is a topological group and  $\Gamma$  is a discrete group, then let  $Rep(\Gamma, G)$  be the collection of group homomorphisms from  $\Gamma$  to  $G$ .  $Rep(\Gamma, G)$  is naturally a subset of the (topological) infinite product  $G^\Gamma$  (which has the product topology). Give  $Rep(\Gamma, G)$  the restriction of the topology from  $G^\Gamma$ . We call this topology on  $Rep(\Gamma, G)$  the *pointwise- $G$  topology*.

**Definition 4.1.** *A topological group  $G$  is said to have the Kirchberg property if every homomorphism from  $\mathbb{F}_\infty \times \mathbb{F}_\infty$  into  $G$  can be approximated, in the pointwise- $G$  topology, by homomorphisms with precompact image.*

The Kirchberg property seems to be a type of amenability property; in particular, it implies amenability. In [15], Pestov and Uspenskij showed that the isometry group of the Urysohn metric space has the Kirchberg property. In the same paper, Pestov and Uspenskij also observed the following equivalences:

**Proposition 4.1.** *The following two statements are equivalent:*

- (1) *The Connes embedding problem has an affirmative answer.*
- (2) *The unitary group  $U(\mathcal{H})$  over a separable infinite dimensional Hilbert space  $\mathcal{H}$ , given the weak\* topology, has the Kirchberg property.*

Following this line of thought, we use our uniqueness theorem (Proposition 3.1) and ideas from the theory of absorbing extensions to show that the Connes embedding problem is equivalent to many topological groups having the Kirchberg property. Firstly, we fix some notation. For the multiplier algebra  $\mathcal{M}(\mathcal{B})$  of a stable  $C^*$ -algebra  $\mathcal{B}$ , let  $U(\mathcal{M}(\mathcal{B}))$  be the unitary group of  $\mathcal{M}(\mathcal{B})$ , given the strict topology. For a von Neumann algebra  $\mathcal{M}$ , let  $U(\mathcal{M})$  be the unitary group of  $\mathcal{M}$ , given the weak\* topology (which, on the unitary group, is the same as the strong topology, weak topology,  $\sigma$ -strong\* topology,  $\sigma$ -weak topology etc.).

**Theorem 4.2.** *The following statements are equivalent:*

- (1) *The Connes embedding problem has an affirmative answer.*
- (2) *For every unital separable simple nuclear  $C^*$ -algebra  $\mathcal{A}$ ,  $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$  has the Kirchberg property.*
- (3) *There exists a unital separable simple nuclear  $C^*$ -algebra  $\mathcal{A}$  such that  $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$  has the Kirchberg property.*
- (4) *For every countably decomposable injective properly infinite von Neumann algebra  $\mathcal{M}$ ,  $U(\mathcal{M})$  has the Kirchberg property.*
- (5) *There exists an injective properly infinite von Neumann algebra  $\mathcal{M}$  with separable predual such that  $U(\mathcal{M})$  has the Kirchberg property.*

*Proof.* That (2) implies (1) follows from Proposition 4.1. That (2) implies (3) and that (4) implies (5) are immediate.

We now prove that (1) implies (2). Let  $\pi : \mathbb{F}_\infty \times \mathbb{F}_\infty \rightarrow U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$  be a group homomorphism.

Recall that the strict topology on  $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  is generated by seminorms of the form  $\|\cdot\|_b$ ,  $b \in \mathcal{A} \otimes \mathcal{K}$ , where for all  $m \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  and for all  $b \in \mathcal{A} \otimes \mathcal{K}$ ,  $\|m\|_b =_{df} \|mb\| + \|bm\|$ .

So let  $\epsilon > 0$  and let a finite subset  $\{x_1, x_2, \dots, x_m\} \subseteq \mathbb{F}_\infty \times \mathbb{F}_\infty$  and a finite subset  $\mathcal{G} \subseteq \mathcal{A} \otimes \mathcal{K}$  be given. (We will work with the seminorms corresponding to elements of  $\mathcal{G}$ ). Contracting  $\epsilon > 0$  if necessary, we may assume that the elements of  $\mathcal{G}$  all have norm less than or equal to one.

Let  $S_1, S_2$  be isometries in  $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  such that the following hold:

(4.1)

- i.  $S_1 S_1^* + S_2 S_2^* = 1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})}$ .
- ii.  $\|S_1 \pi(x_k) S_1^* - \pi(x_k)\|_b < \epsilon/8$  for  $1 \leq k \leq m$ .
- iii.  $\|S_2 S_2^*\|_b < \epsilon/8$

Let  $\mathcal{C} \subset \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  be the unital  $C^*$ -subalgebra that is generated by  $\{\pi(x_1), \pi(x_2), \dots, \pi(x_m)\}$ . Let  $\psi : \mathcal{C} \rightarrow 1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}) \subset \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  be a unital full \*-homomorphism.

Let  $\phi : \mathcal{C} \rightarrow \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  be the unital full  $*$ -homomorphism that is given by  $\phi(c) =_{df} S_1 c S_1^* + S_2 \psi(c) S_2^*$ . Note that by (4.1), we have that

$$(4.2) \quad \|\phi(\pi(x_k)) - \pi(x_k)\|_b < \epsilon/4$$

for  $1 \leq k \leq m$  and  $b \in \mathcal{G}$ .

By [6],  $\phi$  and  $\psi$  are approximately unitarily equivalent (in the pointwise-norm topology). Hence, let  $u \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  be a unitary such that  $\|\phi(\pi(x_k)) - u\psi(\pi(x_k))u^*\| < \epsilon/8$  for  $1 \leq k \leq m$ . From this and (4.2),

$$(4.3) \quad \|\pi(x_k) - u\psi(\pi(x_k))u^*\|_b < \epsilon/2$$

for  $1 \leq k \leq m$  and  $b \in \mathcal{G}$ . But by (1) and Proposition 4.1,  $u(1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}))u^*$  is assumed to have the Kirchberg property. Hence, there is a group homomorphism  $\mu : \mathbb{F}_{\infty} \times \mathbb{F}_{\infty} \rightarrow u(1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}))u^*$  with precompact range such that  $\|\mu(x_k) - u\psi(\pi(x_k))u^*\|_b < \epsilon/2$  for  $1 \leq k \leq m$  and  $b \in \mathcal{G}$ . From this and (4.3), we have that

$$\|\pi(x_k) - \mu(x_k)\|_b < \epsilon$$

for  $1 \leq k \leq m$  and  $b \in \mathcal{G}$ . Since  $\{x_1, x_2, \dots, x_k\}$ ,  $\epsilon$  and  $\pi$  are arbitrary,  $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$  has the Kirchberg property. This completes the proof that (1) implies (2).

Next, we prove that (3) implies (1).

Let  $\mathcal{H}$  be a separable infinite dimensional Hilbert space. By Proposition 4.1, to prove (1), it suffices to prove that  $U(\mathcal{H})$ , given the strong topology, has the Kirchberg property.

Since  $\mathcal{A}$  is a unital separable simple  $C^*$ -algebra, we may assume that  $\mathcal{A} \otimes \mathcal{K}$  is a  $C^*$ -subalgebra of  $\mathbb{B}(\mathcal{H})$  such that

- i.  $\mathcal{A} \otimes \mathcal{K}$  acts nondegenerately on  $\mathcal{H}$ ;
- ii.  $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) = \{m \in \mathbb{B}(\mathcal{H}) : mb, bm \in \mathcal{A} \otimes \mathcal{K}, \forall b \in \mathcal{A} \otimes \mathcal{K}\}$ ; and
- iii. the strict topology on  $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$  is stronger than the strong operator topology from  $\mathbb{B}(\mathcal{H})$ .

(Note that this implies that  $1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} = 1_{\mathbb{B}(\mathcal{H})}$ .)

The rest of the argument is the same as that of (1) implies (2), except that we reverse  $\mathbb{B}(\mathcal{H})$  (or  $1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H})$ ) and  $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ .

The arguments that (1) implies (4) and that (5) implies (1) is the same as that of (1) implies (2) and (3) implies (1) respectively. We note that since  $\mathcal{M}$  is properly infinite,  $\mathcal{M} \cong \mathcal{M} \otimes \mathbb{B}(\mathcal{H})$ . And also, in place of the uniqueness results from [6], we need to use Proposition 3.1.  $\square$

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